AXISYMMETRIC VIBRATIONS OF SIMPLY SUPPORTED CYLINDRICAL SHELLS

(OSESIMMETRICHESKIE KOLEBANIIA SVOBODNO OPERTYKH TSILINDRICHESKIKH OBOLOCHEK)

PMM Vol.25, No.4, 1961, pp. 742-745

K. K. LIVANOV
(Moscow)

(Received April 28, 1961)

The axisymmetric vibrations of cylindrical shells have been investigated in [1,2]. In [1], using the method of undetermined length developed in [3], there is obtained an exact numerical solution of a problem of vibrations of a hollow cylindrical shell with clamped and free supported edges. Experimental work on axisymmetrical vibrations of cylindrical shells was conducted in [2].

In this paper the problem of axisymmetrical vibrations of a cylindrical shell is solved using a displacement function. An expression is found for such a function which gives all characteristic (eigen) functions of the boundary-value problem. The frequencies and the modes of vibrations of the shell for the simply supported edges are investigated, taking into account all inertia forces and also considering only the normal resultant of inertia force, in a wide range of variation of dimensionless curvature. The comparison of the results indicates that the frequency of vibrations, calculated without taking into account the tangential inertia forces, is close to the lower frequency only for small values of dimensionless curvature. It is also shown that the vibrations with extensive transverse displacements do not always have the lowest frequency. Such vibrations have frequencies close to the frequencies calculated without tangential inertia forces.

1. Consider a cylindrical shell of length a, radius R and thickness h. The natural axisymmetrical vibrations of such a shell *in vacuo* are governed by the following equations [4, p. 257]:

$$\begin{gathered}\frac{\partial^2 u}{\partial \alpha^2}+\delta n\left(\nu\frac{\partial w}{\partial \alpha}-c\frac{\partial^2 w}{\partial \alpha^3}\right)=\frac{1-\nu^2}{E}a^2\mu\frac{\partial^2 u}{\partial t^2}\\ \delta n\left(\nu\frac{\partial u}{\partial \alpha}-c\frac{\partial^3 u}{\partial \alpha}\right)+c\frac{\partial^4 w}{\partial \alpha^4}+(c\delta^4 n^4+\delta^2 n^2)w=-\frac{1-\nu^2}{E}a^2\mu\frac{\partial^2 w}{\partial t^2}\end{gathered}\qquad(1.1)$$

where a is the coordinate along the generator of the cylinder; $\delta = a/nR$ is dimensionless curvature of the shell; $c = h^2/12a^2$; E, ν, μ are Young's modulus, Poisson's ratio and density of the material, respectively.

Consider that a shell is executing harmonic vibrations with frequency ω. Expressing the displacements

$$u(\alpha, t) = u(\alpha)\, e^{\omega t}, \qquad w(\alpha, t) = w(\alpha)\, e^{\omega t}$$

we rewrite (1.1) as

$$\begin{gathered}\left(\frac{d^2}{d\alpha^2} - \Omega^2 n^2\right) u + \delta n \left(\nu \frac{d}{d\alpha} - c\frac{d^3}{d\alpha^3}\right) w = 0 \qquad \left(\Omega^2 = \frac{a^2\mu(1-\nu^2)}{n^2 E}\ \omega^2\right) \\ \delta n \left(\nu \frac{d}{d\alpha} - c\frac{d^3}{d\alpha^3}\right) u + \left(c\frac{d^4}{d\alpha^4} + \Omega^2 n^2 + c\delta^4 n^4 + \delta^2 n^2\right) w = 0\end{gathered} \tag{1.2}$$

where Ω is a dimensionless frequency. In (1.1) and (1.2) n is an arbitrary number, which will be taken to be equal to the number of longitudinal half-waves of the shell.

Consider now the operational determinant in (1.2). In order to introduce the displacement function it is necessary to consider algebraic complements either of the elements of the first line or of the elements of the second line of this determinant.

2. Let us introduce a displacement function $\Phi(a)$ by the following:

$$u(\alpha) = \delta n\,(c\Phi''' - \nu\Phi'), \qquad w(\alpha) = \Phi'' - \Omega^2 n^2 \Phi \tag{2.1}$$

Substituting (2.1) into (1.2), we see that the first equation of system (1.2) is identically satisfied, and the second one has the form

$$\begin{gathered} c(1 - \delta^2 n^2 c)\,\Phi^{\mathrm{VI}} + cn^2(2\delta^2\nu - \Omega^2)\,\Phi^{\mathrm{IV}} + \\ + n^2(\Omega^2 + \delta^4 n^2 c + \delta^2 - \delta^2\nu^2)\,\Phi'' - n^4\Omega^2(\Omega^2 + \delta^4 n^2 c + \delta^2)\,\Phi = 0 \end{gathered} \tag{2.2}$$

Substituting (2.1) into the expressions for stresses and moments in terms of the displacements [4, p. 256], we obtain

$$N_1 = \frac{Eh\Omega^2 n^2}{(1-\nu^2)R}(c\,\Phi'' - \nu\Phi)$$

$$M_1 = \frac{Eh^3}{12(1-\nu^2)a^2}\left[(1 - \delta^2 n^2 c)\,\Phi^{\mathrm{IV}} + (\delta^2 n^2\nu - \Omega^2 n^2)\,\Phi''\right]$$

In the case of movable hinges in the axial direction along the edges of a shell we have

$$w = 0, \qquad N_1 = 0, \qquad M_1 = 0 \text{ for } \alpha = 0,\ \alpha = 1 \tag{2.3}$$

Since $(1 - \delta^2 n^2 c) > 0$ (the thickness of the shell is smaller than the radius) and $c\Omega^2 n^2 - \nu \neq 0$ (Ω in a given problem cannot be a real number if we disregard the rigid body motion of the shell) then (2.3) can be written as

$$\Phi = \Phi'' = \Phi^{IV} = 0 \quad \text{for } \alpha = 0, \quad \alpha = 1 \tag{2.4}$$

Solution of (2.2) has the form $\Phi(\alpha) = c_1 e^{z_1 \alpha} + \ldots + c_6 e^{z_6 \alpha}$, where $z_1 \ldots z_6$ are the roots of the characteristic polynomial

$$c(1 - \delta^2 n^2 c) z^6 + cn^2 (2\delta^2 \nu - \Omega^2) z^4 + \\ + n^2 (\Omega^2 + \delta^4 n^2 c + \delta^2 - \delta^2 \nu^2) z^2 - n^4 \Omega^2 (\Omega^2 + \delta^4 n^2 c + \delta^2) = 0 \tag{2.5}$$

Utilizing now the boundary conditions (2.4) it is possible to construct out of the roots z_k a characteristic determinant and equate it to zero. The form of this determinant depends upon the multiplicity of the roots. A.A. Movchan demonstrated, however, that it is possible to avoid the consideration of the different cases of root multiplicity by considering the equations

$$\Delta / \sigma = 0 \tag{2.6}$$

where Δ is a characteristic determinant for single roots, σ is Vandermonde's determinant formed with the values $z_1 \ldots z_6$. Since the characteristic polynomial contains only even powers of z, we can write $z_1 = -z_4$, $z_2 = -z_5$, $z_3 = -z_6$. Thus we have

$$\frac{\sinh z_1}{z_1} \frac{\sinh z_2}{z_2} \frac{\sinh z_3}{z_3} = 0$$

It is clear then that the boundary conditions (2.4) are satisfied if the characteristic roots of (2.5) are $\pi n i$ or $-\pi n i$ $(n = 1, 2, 3 \ldots)$. Thus solutions of (2.2) satisfying (2.4) are

$$\Phi(\alpha) = \sin \pi n \alpha \tag{2.7}$$

It can be shown that all other solutions of the boundary-value problem (2.2), (2.4) differ from (2.7) by a multiplicative constant. Substituting $z = \pi n i$ in (2.5), we obtain the frequency equation for a simply supported shell

$$\Omega^4 + [\pi^2 + \delta^2 + cn^2 (\pi^4 + \delta^4)] \Omega^2 + \pi^2 \delta^2 (1 - \nu^2) + \\ + cn^2 [(1 - \delta^2 n^2 c) \pi^6 + \pi^2 \delta^4 - 2\delta^2 \nu \pi^4] = 0 \tag{2.8}$$

For fixed values of δ, n, c, ν, (2.8) yields four frequencies

$$\Omega_1 = i \{0{,}5 [\pi^2 + \delta^2 + cn^2 (\pi^4 + \delta^4)] - 0.5 ([\pi^2 + \delta^2 + cn^2 (\pi^4 + \delta^4)]^2 - \\ - 4\delta^2 \pi^2 (1 - \nu^2) - 4cn^2 [(1 - \delta^2 cn^2) \pi^6 - 2\delta^2 \pi^4 \nu + \pi^2 \delta^4])^{1/2}\}^{1/2}$$

$$\Omega_2 = i \{0.5 [\pi^2 + \delta^2 + cn^2 (\pi^4 + \delta^4)] + 0.5 ([\pi^2 + \delta^2 + cn^2 (\pi^4 + \delta^4)]^2 - \\ - 4\delta^2 \pi^2 (1 - \nu^2) - 4cn^2 [(1 - \delta^2 cn^2) \pi^6 - 2\delta^2 \pi^4 \nu + \pi^2 \delta^4])^{1/2}\}^{1/2}$$

$$\Omega_3 = -\Omega_1, \qquad \Omega_4 = -\Omega_2 \tag{2.9}$$

For $\delta = 0$ (for a strip of width a) we obtain $\Omega_1 = \sqrt{c\pi^2 n i}$, $\Omega_2 = \pi i$.

Substituting (2.7) into (2.1), we obtain the displacements

$$u(\alpha) = -\delta\pi n^2 (\nu + \pi^2 n^2 c)\cos \pi n\alpha, \qquad w(\alpha) = -n^2(\pi^2 + \Omega^2)\sin \pi n\alpha \quad (2.10)$$

3. It is also possible to introduce the displacement function in another way

$$u(\alpha) = c\Phi^{\mathrm{IV}} + n^2(\Omega^2 + \delta^4 n^2 c + \delta^2)\Phi, \qquad w(\alpha) = \delta n(c\Phi''' - \nu\Phi') \quad (3.1)$$

In this case the second equation of system (1.2) is satisfied identically and the first one reduces to (2.2). Expressing the stress resultant and moment by the displacement function, the boundary conditions (2.3) can be expressed as follows:

$$\Phi' = \Phi''' = \Phi^{\mathrm{V}} = 0 \quad \text{for } \alpha = 0,\ \alpha = 1 \quad (3.2)$$

Thus the equation will be reduced to

$$z_1 z_2 z_3 \sinh z_1 \ \sinh z_2 \ \sinh z_3 = 0$$

We find then $z_k = \pm \pi n i$. Thus the solution of (2.2) satisfying (3.2) has the form

$$\Phi(\alpha) = \cos \pi n\alpha \quad (3.3)$$

The frequency equation coincides with (2.8). The displacements are obtained by substituting (3.3) into (3.1)

$$u(\alpha) = n^2(\Omega^2 + \delta^2 + \delta^4 n^2 c + \pi^4 n^2 c)\cos \pi n\alpha, \qquad w(\alpha) = \delta n^2 \pi(\nu + c\pi^2 n^2)\sin \pi n\alpha \quad (3.4)$$

To simplify calculations one neglects sometimes the inertia forces due to the tangential displacement. This assumption is equivalent to dropping the time derivative in the first equation of (1.1). As a consequence of this, in the first equation in (1.2) and in (2.1) the terms which are multiplied by the frequency vanish. (2.8) now is

$$\Omega^2 + \delta^2(1 - \nu^2) + n^2 c(\delta^4 + \pi^4 - 2\delta^2\nu\pi^2 - \pi^4\delta^2 n^2 c) = 0 \quad (3.5)$$

The frequency thus determined will be denoted by Ω^-. For $\delta = 0$, $\Omega^- = \Omega_1$.

4. To estimate the influence of the nondimensional curvature δ on the frequencies Ω_1, Ω_2 and Ω^-, the results of numerical calculations for $c = 1/3 \times 10^{-6}$; $\nu = 1/3$; $n = 1, 2 \ldots 5$; $\delta = 1, 2 \ldots 20$ are plotted in Fig. 1. From this figure it is seen that Ω^- and Ω_1 are close for small values of δ ($\delta < 2$). For large δ, Ω^- differs considerably from Ω_1.

Neglecting in (2.9) and (3.6) terms multiplied by small parameter c, we obtain the following approximate frequency formulas:

$$\Omega_1 = i\sqrt{0.5\,(\pi^2 + \delta^2 - \sqrt{(\pi^2 - \delta^2)^2 + 4\delta^2\nu^2\pi^2})}$$

$$\Omega_2 = i\sqrt{0.5\,(\pi^2 + \delta^2 + \sqrt{(\pi^2 - \delta^2)^2 + 4\delta^2\nu^2\pi^2})}$$

$$\Omega^- = i\delta\sqrt{1 - \nu^2}$$

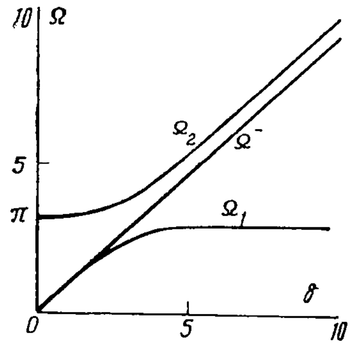

Fig. 1.

For $c = 1/3 \times 10^{-6}$; $\nu = 1/3$; $n < 6$; $1 \leqslant \delta \leqslant 20$, these formulas yield at least three correct figures of the frequencies.

Consider now displacements of the cylindrical shell. For each frequency Ω_1 or Ω_2, (2.10) gives two systems of displacements. Since the frequencies satisfy (2.8), it is easy to demonstrate, however, that these displacements are proportional to each other. Thus, for the determination of the displacements corresponding to a given frequency one may use either formula in (2.10) or (3.4). For the sake of definiteness we shall use (2.10) for Ω_1, and (3.4) for Ω_2. It is easy then to show that using the identity

$$\Omega_2^2 + \delta^2 + \delta^4 n^2 c + \pi^4 n^2 c \equiv -(\Omega_1^2 + \pi^2)$$

we obtain the following two systems of displacements:

$$u(\alpha) = P\cos\pi n\alpha, \qquad w(\alpha) = \sin\pi n\alpha \tag{4.1}$$

$$u(\alpha) = \cos\pi n\alpha, \qquad w(\alpha) = -P\sin\pi n\alpha, \qquad P = \frac{\delta\pi(\nu + \pi^2 n^2 c)}{\Omega_1^2 + \pi^2} \tag{4.2}$$

The displacements (4.1) correspond to the frequency Ω_1 and (4.2) correspond to Ω_2.

The magnitude P we shall call amplitude. The amplitude P was calculated for $c = 1/3 \times 10^{-6}$; $\nu = 1/3$; $n = 1, \ldots, 5$; $\delta = 0, 1, \ldots, 20$. The calculations show that the changes in the value of n influence only the third figure in the amplitude. For this reason in Fig. 2 the relationship between P and δ is represented by one line for various n. An approximate formula for P is obtained by dropping terms multiplied by the small factor c.

$$P = \frac{2\delta\nu\pi}{\pi^2 - \delta^2 + \sqrt{(\pi^2 - \delta^2)^2 + 4\delta^2\nu^2\pi^2}} \tag{4.3}$$

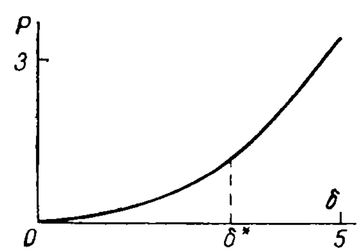

Fig. 2.

Formula (4.3) yields at least three correct figures for $c \leqslant 1/3 \times 10^{-6}$;

$n < 6$; $\nu = 1/3$; $0 \leqslant \delta \leqslant 20$.

Let us now investigate the dependence of P on δ. For $\delta = 0$ (for an infinite strip), $P = 0$. Using (4.1) and (4.2) we see that, for $\delta = 0$, Ω_1 is the transverse frequency ($w \neq 0$) and Ω_2 is the longitudinal frequency ($u \neq 0$). For $\delta > 0$, P is different from zero. This means that to each frequency there will correspond vibrations with nonzero transverse and longitudinal displacements. If $P < 1$, we shall call Ω_1 the frequency of predominantly transverse vibrations and Ω_2 the frequency of predominantly longitudinal vibrations; the nomenclature will be reversed if $P > 1$. From the graph in Fig. 2 it is seen that if $\delta < \delta^*$ (δ^* is the value of curvature for $P = 1$; it depends on c, n and ν) then Ω_1 will be the frequency of predominantly transverse vibrations, and Ω_2 of predominantly longitudinal vibrations. For $\delta > \delta^*$, Ω_1 (lower for a given set of parameters) will be the frequency of predominantly longitudinal vibrations, and Ω_2 will be the frequency of predominantly transverse vibrations. An approximate value of δ^* for a few first values of n may be found by setting $P = 1$ in (4.3). Solving for δ we obtain $\delta^* = \pi$.

It is seen that the lower frequency does not always correspond to predominantly transverse vibrations [5, p. 118], [6, p. 135]. Thus, for the range of parameters under consideration, the lower frequency corresponds to the longitudinal vibrations provided that $\delta > \delta^*$. It has to be noted that Ω^- is close to Ω_1 for small δ, and to Ω_2 for large δ. In other words, Ω^- is always close to the frequency of predominantly transverse vibrations.

In the experiments [2] the shell was given initial transverse excitations, and consequently the frequency of predominantly transverse vibrations was measured there, that is, the frequency close to Ω^-. It was demonstrated above, however, that this "flexural frequency" need not be always the lowest frequency. For large δ, the frequency of predominantly longitudinal vibrations will be substantially lower than the "flexural frequency" observed experimentally.

BIBLIOGRAPHY

1. Ollik, K.K. Ob osesimmetricheskikh kolebaniiakh krugotsilindricheskikh obolochek (On axisymmetrical vibrations of cylindrical shells). *Tr. Tallinskogo politekhnichesk. in-ta* No. 147, 1958.

2. Breslavskii, V.E., O kolebaniiakh tsilindricheskikh obolochek (On vibrations of cylindrical shells). *Inzh. sb. Akad. Nauk SSSR* Vol. 16, 1953.

3. Flügge, W., *Statik und Dynamik der Schalen*. Berlin, 1934.

4. Vlasov, V.Z., *Obshchaia teoriia obolochek* (*General Theory of Shells*). Gostekhizdat, 1949.

5. Breslavskii, V.E., Sobstvennye kolebaniia krugovoi tsilindricheskoi obolochki, nakhodiashcheisia pod deistviem gidrostaticheskogo davleniia (Natural vibrations of cylindrical shells under hydrostatic pressure). *Izv. Akad. Nauk SSSR, OTN* No. 12, 1956.

6. Nikulin, M.M., Vliianie osevykh usilii na chastoty sobstvennykh kolebanii tsilindricheskoi obolochki (Influence of axial forces on the natural frequencies of cylindrical shells). *Sb: Prochnost tsilindrichskikh obolochek* (*Collection: Strength of Cylindrical Shells*). Oborongiz, 1959.

Translated by R.M. E.-I.